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Interpolation between L^∞ and H^1, the real method*

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The intermediate spaces in the Lions–Peetre method of interpolation, between the class of continuous functions vanishing at infinity and the H^1 class, are calculated. The result implies identification of intermediate spaces between the class of all finite Borel measures and the class of functions of bounded mean oscillation.

INTRODUCTION

In [2], Fefferman and Stein have identified the intermediate spaces between H^1 and L^p, in the complex method of interpolation. They have shown that

$$[H^1, L^{p_1}]_\theta = L^p, \tag{1}$$

where $1/p = (1 - \theta)/1 + \theta/p_1$, $1 < p_1 \leqslant \infty$. In this note we show that in the real method one gets

$$(H^1, L(p_1, q_1))_{\theta,q} = L(p, q) \tag{2}$$

with $1/p = (1 - \theta)/1 + \theta/p_1$, $1 < p_1 \leqslant \infty$. In particular, if $q_1 = p_1$, $p = q$, one gets

$$(H^1, L^{p_1})_{\theta,p} = L^p. \tag{3}$$

Our results imply recent results of Stroock [8], who proved, using probabilistic methods, that if

$$T\colon H^1 \to L(1, \infty),$$

$$T\colon L^2 \to L(2, \infty),$$

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then

$$T: L^p \to L^p.$$

Actually, we prove

$$(H^1, C_\omega)_{\theta,q} = L(p, q) \qquad p = 1/(1-\theta), \tag{4}$$

where C_ω is the class of continuous functions tending to 0 at infinity. This clearly implies (3) for $p_1 = q_1 = \infty$, which in turn implies all other results.

By a duality argument, we also get for $0 < q \leqslant \infty$, $0 < \theta < 1$

$$(BMO, M)_{1-\theta,q} = L(p, q). \tag{5}$$

To make the presentation reasonably self-contained, we shall include a resumé of the pertinent notions of real interpolation theory in section 1. For a more detailed exposition and for proofs of the theorems quoted, the reader is referred to [3–5].

1. Real Interpolation

Let (A_0, A_1, A) be an interpolation triple, i.e., A_i are quasinormed spaces, continuously embedded in A.

Definition 1. Let $a \in A_0 + A_1$. Define

$$K(t, a) = K(t, a; A_0, A_1; A) = \inf\{\max_i t^i \| a_i \|_{A_i} \mid a_0 + a_1 = a\}.$$

$$\| a \|_{\theta,q} = \left(\int_0^\infty [t^{-\theta} K(t, a)]^q \, dt/t\right)^{1/q},$$

where $0 < \theta < 1$, $0 < q \leqslant \infty$.

$$(A_0, A_1)_{\theta,q} = \{a \in A_0 + A_1 \mid \| a \|_{\theta,q} < \infty\}.$$

One easily checks that $((A_0, A_1)_{\theta,q}, \| \|_{\theta,q})$ is a quasinormed space.

Definition 2. $(A_0, A_1; A)$, $(B_0, B_1; B)$ are two interpolation triples.

$$T: A_0 + A_1 \to B_0 + B_1$$

is a quasilinear map from $(A_0, A_1; A)$ to $(B_0, B_1; B)$ iff given

$a \in A_0 + A_1$ and $a_i \in A_i$ with $a_0 + a_1 = a$, then $b_i \in B_i$ can be found satisfying

(1) $Ta = b_0 + b_1$,

(2) $\| b_i \|_{B_i} \leqslant K_i \| a_i \|_{A_i}$.

THEOREM 3. *T is a quasilinear map from $(A_0, A_1; A)$ to $(B_0, B_1; B)$. Then*

$$T: (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$$

and

$$\| Ta \|_{B,\theta,q} \leqslant K_0^{1-\theta} K_1^{\theta} \| a \|_{A,\theta,q}.$$

THEOREM 4. *$(A_0, A_1; A)$ is an interpolation triple. $E_i = (A_0, A_1)_{\theta_i, q_i}$, $\theta_0 \neq \theta_1$. Then*

$$(E_0, E_1)_{\lambda,q} = (A_0, A_1)_{\theta,q},$$

where $\theta = (1 - \lambda)\,\theta_0 + \lambda\theta_1$. Also,

$$(A_0, E_1)_{\lambda,q} = (A_0, A_1)_{\lambda\theta_1,q}.$$

THEOREM 5. *$(A_0, A_1; A)$ is an interpolation triple, A_i Banach spaces, and $A_0 \cap A_1$ is dense in A_i, then for $1 \leqslant q < \infty$*

$$(A_0, A_1)'_{\theta,q} = (A_0', A_1')_{\theta,q'}.$$

(A_i' is the dual of A_i, and $1/q + 1/q' = 1$.)

We next turn to $L(p, q)$ spaces. (X, Σ, μ) is a σ-finite measure space. B is a Banach space. f is a strongly measurable B-valued function on X.

DEFINITION 6. $f_*(y) = \mu\{x \mid |f(x)| > y\}$ ($|\ |$ is the B-norm).

$$f^*(t) = \inf\{y \mid f_*(y) \leqslant t\}.$$

Define, also,

$$\| f \|_{p,q} = \begin{cases} \left(\int_0^\infty t^{q/p} [f^*(t)]^q \, dt/t \right)^{1/q} & \begin{matrix} 0 < p < \infty \\ 0 < q < \infty. \end{matrix} \\ \sup t^{1/p} f^*(t) & \begin{matrix} 0 < p \leqslant \infty \\ q = \infty. \end{matrix} \end{cases}$$

$$L(p, q) = \{f \text{ strongly measurable} \mid \| f \|_{p,q} < \infty\}.$$

Note: $L(p, p) = L^p$. $L(p, \infty) = L^p$ weak.

THEOREM 7. *$(L(p_0, q_0), L(p_1, p_1))_{\theta,q} = L(p, q)$, where*

$$1/p = (1-\theta)/p_0 + \theta/p_1 \quad p_0 \neq p_1 \quad 0 < p_i, q_i \leqslant \infty.$$

We shall need in the sequel also the following inequality, which is a variation on one by Hardy.

THEOREM 8. *Let $0 \leqslant f$ be monotone nonincreasing on $(0, \infty)$. $0 < q \leqslant \infty$, $0 < r < q$. Then*

$$\left(\int_0^\infty \left(1/t \int_0^t f(s)\, ds\right)^q t^r\, dt/t\right)^{1/q} \leqslant C_{q,r} \left(\int_0^\infty (f(t))^q t^r\, dt/t\right)^{1/q}.$$

Proof. Consider the sublinear opeator

$$(Tf)(t) = 1/t \int_0^t |f(s)|\, ds.$$

By Hölder's inequality

$$Tf(t) \leqslant t^{-1/p} \|f\|_{p,p},$$

so that

$$(Tf)^*(t) \leqslant t^{-1/p} \|f\|_{p,p},$$

and

$$T: L(p, p) \to L(p, \infty) \qquad 1 \leqslant p \leqslant \infty.$$

Interpolating, we get for $1 < p < \infty$, $0 < q \leqslant \infty$,

$$T: L(p, q) \to L(p, q),$$

i.e.,

$$\left(\int_0^\infty \left[\left(1/u \int_0^u f\right)^* (t)\right]^q t^{q/p}\, dt/t\right)^{1/q} \leqslant C_{q,p} \left(\int_0^\infty [f^*(t)]^q t^{q/p}\, dt/t\right)^{1/q}.$$

If now f is nonincreasing, so is $1/u \int_0^u f$, so that $f^*(t) = f(t)$ and $(1/u \int_0^u f)^*(t) = 1/t \int_0^t f$. Substituting $q/p = r$. We get the desired inequality. For an inequality analogous to the above, see [6].

2. Interpolation of H^1 and C_ω, the real method

Let R_j be the jth Riesz transform

$$R_j(f)(x) = p \cdot v \cdot \int_{R^n} f(x-y)\, y_j/|\, y\,|^{n+1}\, dy.$$

H^1 is the subspace of L^1 defined by the norm (see [7])

$$\|f\|_{H^1} = \|f\|_{L^1} + \sum_{j=1}^{n} \|R_j f\|_{L^1}.$$

Definition 1. I are cubes.

$$M_r(f)(x) = \sup_{x \in I} \left(1/|\,I\,| \int_I |f|^r\right)^{1/r},$$

$$f_r(t) = \left(1/t \int_0^t [M_r(f)^*(s)]^r\, ds\right)^{1/r}.$$

Lemma 2. *f is supported on a cube I and has average* 0 *on it. Then, for* $1 < r < \infty$,

$$\|f\|_{H^1} \leqslant C \cdot |\,I\,|^{1-1/r} \|f\|_r.$$

Proof. Clearly $\|f\|_{L^1} \leqslant |\,I\,|^{1-1/r} \|f\|_r$. Denote by $2I$ the symmetric dilation of I,

$$\int_{2I} |\,R_j f\,| \leqslant C \cdot |\,I\,|^{1-1/r} \left(\int_{2I} |\,R_j f\,|^r\right)^{1/r}$$
$$\leqslant C \cdot |\,I\,|^{1-1/r} \left(\int_I |\,f\,|^r\right)^{1/r}.$$

Denote by $\bar{y}$ the center of I.

$$\int_{(2I)'} |\,R_j f\,| \leqslant \int_{(2I)'} \int_I |f(y)| \left| \frac{x_j - y_j}{|\,x-y\,|^{n+1}} - \frac{x_j - \bar{y}_j}{|\,x-\bar{y}\,|^{n+1}} \right| dy\, dx$$
$$= \int_I |f(y)| \int_{(2I)'} \left| \frac{x_j - y_j}{|\,x-y\,|^{n+1}} - \frac{x_j - \bar{y}_j}{|\,x-\bar{y}\,|^{n+1}} \right| dx\, dy$$
$$\leqslant C \|f\|_{L^1} \leqslant C\, |\,I\,|^{1-1/r} \|f\|_r,$$

and so

$$\|\,R_j f\,\|_1 \leqslant C \cdot |\,I\,|^{1-1/r} \|f\|_r.$$

LEMMA 3. *f is supported on a cube I. $1 < r < \infty$.*

$$\left(1/|I| \int_I |f|^r\right)^{1/r} = a.$$

Then

$$K(t, f; H^1, C_\omega) \leqslant Ca(t + |I|).$$

Proof. Let $0 \leqslant \phi \leqslant 1$ be continuous, $\phi = 1$ on I, and vanish outside $2I$. Write

$$f = f - b\phi + b\phi,$$

where

$$\int (f - b\phi)\, dx = 0, \qquad |b| \leqslant a.$$

$$K(t, f) \leqslant \| f - b\phi \|_{H^1} + t \| b\phi \|_{C_\omega} \leqslant Ca(t + |I|).$$

THEOREM 4. *$(H^1, C_\omega)_{\theta,q} = L(p, q)$, where*

$$0 < q \leqslant \infty, \qquad p = 1/(1 - \theta).$$

Proof. Clearly $(H^1, C_\omega)_{\theta,q} \subset (L^1, L^\infty)_{\theta,q} = L(p, q)$. We shall show for $f \in C_\omega$,

$$\| f \|_{(H^1, C_\omega)_{\theta,q}} \leqslant C \cdot \| f \|_{p,q}.$$

Choose $0 < t$, $1 < r < p$. Apply a Calderón–Zygmund decomposition of R^n (see [1]) to get a sequence of cubes $\{I_j\}$ with disjoint interiors, so that

$$f_r(t) < \left(\frac{1}{|I_j|} \int_{I_j} |f|^r\right)^{1/r} \leqslant 2^n f_r(t)$$

and $|f(x)| \leqslant f_r(t)$ at a.e. $x \notin \cup I_j$. Let

$$a_j(f) = \frac{1}{|I_j|} \int_{I_j} f, \qquad f^t = \sum_j (f - a_j(f))\, X_{I_j}.$$

$$\| f^t \|_{H^1} \leqslant C \cdot \sum_j | I_j |^{1-1/r} \| (f - a_j(f))\, X_{I_j} \|_{L^r}$$

$$\leqslant C \cdot \sum_j f_r^{1-r}(t) \, \| f X_{I_j} \|_r^{r-1} \, \| f X_{I_j} \|_r$$

$$= C \cdot f_r^{1-r}(t) \int_{\cup I} |f|^r.$$

Let $E = \{x \mid M_r(f) > f_r(t)\}$. $E^* = \{x \mid M_r(f) > M_r(f)^*(t)\}$. Clearly $\cup I_j \subset E \subset E^*$, and

$$\|f^t\|_{H^1} \leqslant C \cdot f_r^{1-r}(t) \int_{E^*} M_r{}^r(f)\, dx \leqslant C \cdot f_r^{1-r}(t) \int_0^t [M_r(f)^*(s)]^r\, ds$$
$$\leqslant C \cdot tf_r(t).$$

Clearly $\|f - f^t\|_{L^\infty} \leqslant C \cdot f_r(t)$. The support of $f - f^t$ is contained in some cube I (which depends on f). There exists $\epsilon > 0$ so that if $|E| < \epsilon$,

$$\left(1/|I| \int_I |f \cdot X_E|^r\right)^{1/r} < tf_r(t)/(t + |I|).$$

Write now $f - f^t = g + h$, where $g \in C_\omega$, $|g| \leqslant C \cdot f_r(t)$, and the measure of the support of h is $< \epsilon$. By Lemma 3,

$$K(t, h; H^1, C_\omega) < C \cdot tf_r(t).$$

To sum up,

$$f = f^t + g + h.$$

So

$$K(t, f) \leqslant K(t, f^t + g) + K(t, h)$$
$$\leqslant \|f^t\|_{H^1} + t\|g\|_{C_\omega} + C \cdot tf_r(t)$$
$$\leqslant C \cdot tf_r(t).$$

We, therefore, have

$$\|f\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta}K(t,f))^q\, dt/t\right)^{1/q}$$
$$\leqslant C \cdot \left(\int_0^\infty t^{q/p} \left(1/t \int_0^t (M_r(f)^*(s))^r\, ds\right)^{q/r} dt/t\right)^{1/q}.$$

So that by Lemma 1.8 (which we need for the case $q < r$—otherwise we simply use Hardy's inequality), we have

$$\|f\|_{\theta,q} \leqslant C \left(\int_0^\infty t^{q/p}(M_r(f)^*(t))^q\, dt/t\right)^{1/q}$$
$$= C \cdot \|M_r(f)\|_{p,q} \leqslant C\|f\|_{p,q}.$$

Since C_ω is dense in $L(p, q)$, $q < \infty$, the theorem is proved for $q < \infty$. For $q = \infty$ we take $0 < \theta_0 < \theta < \theta_1 < 1$.

We have

$$L^{p_i} = (H^1, C_\omega)_{\theta_i, p_i},$$

with $p_i = 1/(1 - \theta_i)$. We, therefore, have, using Theorem 1.4 (the reiteration theorem)

$$(H^1, C_\omega)_{\theta, \infty} = (L^{p_0}, L^{p_1})_{\lambda, \infty},$$

where $\theta = (1 - \lambda)\, \theta_0 + \lambda\theta_1$. However,

$$(L^{p_0}, L^{p_1})_{\lambda, \infty} = L(p, \infty),$$

with $1/p = (1 - \lambda)/p_0 + \lambda/p_1 = 1 - \theta$. The theorem is proved. Using the reiteration theorem and Theorem 2.4, we have the following theorem.

THEOREM 5. $(H^1, L(p_1, q_1))_{\theta, q} = L(p, q)$, *where*

$$1/p = 1 - \theta + \theta/p_1 .$$

Denote by BMO the space of functions of bounded mean oscillation, by M the space of finite Borel measures. We have: $(H^1)' = BMO$ (see [2]) and $(C_\omega)' = M$. We have the following theorem.

THEOREM 6. $(BMO, M)_{\theta, q} = L(p', q)$, *where*

$$p = 1/(1 - \theta) \qquad 0 < q \leqslant \infty.$$

Proof. We first apply the duality theorem 1.5 to get for $1 < q < \infty$, $p = 1/(1 - \theta)$

$$\begin{aligned}(BMO, M)_{\theta, q} &= ((H^1)', C_\omega')_{\theta, q} = ((H^1, C_\omega)_{\theta, q'})' \\ &= (L(p, q'))' = L(p', q).\end{aligned}$$

Next we use reiteration to get the full result: Given $0 < \theta < 1$, $0 < q \leqslant \infty$, choose $0 < \theta_0 < \theta < \theta_1 < 1$, $1 < q_i < \infty$. We have

$$\begin{aligned}(BMO, M)_{\theta, q} &= ((BMO, M)_{\theta_0, q_0}, (BMO, M)_{\theta_1, q_1})_{\lambda, q} \\ &= (L(p_0', q_0), L(p_1', q_1))_{\lambda, q} = L(p', q).\end{aligned}$$

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